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SOME AMENDMENTS AND CORRECTIONS TO VARIATIONAL PRINCIPLES IN NONLINEAR VISCOELASTICITY

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1. INTRODUCTION

In [1] we discussed variational principles in nonlinear viscoelasticity. The principal result there, was the development of a functional $K(\Lambda)$ which produced as Euler equations the (transformed) field equations of a general theory of nonlinear viscoelasticity. Unfortunately, the functional actually given in [1] will not yield the correct Euler equations unless the stress and dissipation functions $\mathfrak{F}^{ij}$ and $\mathscr{D}$ are held constant when the total strain histories are varied. This limitation invalidates theorem 2 in [1] and makes the collection of results given there considerably less useful than originally reported.

In this note, we again address the variational problem and offer modifications of our earlier work which overcomes the objectionable features mentioned above.

2. A VARIATIONAL THEOREM

Using the notation, and assumed initial- and boundary conditions given in [1], we consider the behavior of a nonlinearly viscoelastic solid for which the following field equations hold at particle $\mathbf{x}$ and time t:

$$\begin{gathered}[\sigma^{ij}(u_{m,j}+\delta_{mj})]_{,i}+\rho F_m=\rho\ddot{u}_m\\ \gamma_{ij}=\tfrac{1}{2}(u_{i,j}+u_{j,i}+u_{m,i}u_{m,j})\\ \sigma^{ij}=\overset{\infty}{\underset{s=0}{\mathfrak{F}^{ij}}}[\boldsymbol{\gamma}^t(s)]\end{gathered}\tag{2.1}$$

where $\boldsymbol{\gamma}^t(s)\equiv\boldsymbol{\gamma}(\mathbf{x},t-s)$ is the history of the strain $\gamma(\mathbf{x},t)$, and $\overset{\infty}{\underset{s=0}{\mathfrak{F}^{ij}}}[\cdot]$ is the stress response functional.

An alternate form of (2.1) can be obtained by taking the Laplace transform of each equation. Under appropriate (homogeneous) initial conditions, the transformed equations assume the form

$$\begin{gathered}\frac{1}{2\pi j}(\bar{\sigma}^{ij}*\bar{u}_{m,j})_{,i}+\bar{\sigma}^{im}_{,i}+\rho\bar{F}_m=\rho p^2\bar{u}_m\\ \bar{\gamma}_{ij}=\tfrac{1}{2}\left[\bar{u}_{i,j}+\bar{u}_{j,i}+\frac{1}{2\pi j}(\bar{u}_{m,i}*\bar{u}_{m,j})\right]\\ \bar{\sigma}^{ij}=F^{ij}[\bar{\boldsymbol{\gamma}}(p),p].\end{gathered}\tag{2.2}$$

IJSS Vol. 10 No. 5—A

Here $\bar{\sigma}^{ij}$ is the Laplace transform of σ^{ij}, etc. in the sense that the Laplace transformation on the time variable of a function $f(\mathbf{x}, t)$ is given by

$$\bar{f}(\mathbf{x}, p) = \int_0^\infty f(\mathbf{x}, t)\bar{e}^{pt}\, dt.$$

Then the inverse transformation of $\bar{f}(\mathbf{x}, p)$ back to the real time is defined by (see, e.g. Churchill[3])

$$f(\mathbf{x}, t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \bar{f}(\mathbf{x}, p)e^{pt}\, dp.$$

Furthermore, in [3] the Laplace transformation of the product of two functions $f_1(t)$ and $f_2(t)$ is given by

$$\mathscr{L}\{f_1(t)f_2(t)\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \bar{f}_1(p-w)\bar{f}_2(w)\, dw$$

or equivalently

$$= \frac{1}{2\pi j}(\bar{f}_1(p) * \bar{f}_2(p))$$

wherein the symbol $*$ denotes the convolution operation. In (2.2) the dependence of $\bar{\sigma}^{ij}$, $\bar{\gamma}_{ij}$ etc. on $\mathbf{x}$ is understood.

In the following we shall make use of the methodology described in [1] to construct the variational statement of the problem described by (2.2).

We shall henceforth assume that the Laplace transformation of the stress functional $\overset{\infty}{\mathscr{F}}{}^{ij}[\gamma^t(s)]$ of (2.1) is defined by an ordinary function of the type $F^{ij}(\boldsymbol{\alpha}(p), p)$. This assertion, of course, may severely limit the forms of $\mathscr{F}$ that can be used in our variational principle. We cite an example of such a functional subsequently. (See equation 3.2.)

Theorem 1

Let $\bar{u}^0$, $\bar{\boldsymbol{\gamma}}^0$ and $\bar{\boldsymbol{\sigma}}^0$, all defined for $(\mathbf{x}, p)\varepsilon\mathscr{U} \times (-\infty, \infty)$, satisfy the field equations (2.2) with homogeneous boundary conditions Further, let $\psi[\bar{\boldsymbol{\gamma}}(\mathbf{x}, p), p]$ denote a functional whose Gateaux differential is such that $\delta\psi[\cdot] \equiv F^{ij}[\bar{\boldsymbol{\gamma}}(\mathbf{x}, p), p] * \hat{\bar{\gamma}}_{ij}$, where $\hat{\bar{\gamma}}_{ij}$ is arbitrary. Then the functional $K(\bar{\boldsymbol{\Lambda}})$ given by

$$K(\bar{\boldsymbol{\Lambda}}) = \tfrac{1}{2}\int_v \{\rho p^2 \bar{u}_m * \bar{u}_m - 2\rho \bar{F}_m * \bar{u}_m - \frac{1}{2\pi j}(\bar{\sigma}^{ij} * \bar{u}_{m,j})_{,i} * \bar{u}_m$$
$$- 2\bar{\sigma}^{im}_{,i} * \bar{u}_m + 2\psi[\bar{\boldsymbol{\gamma}}(p), p] - 2\bar{\gamma}_{ij} * \bar{\sigma}^{ij}\}\, dv \quad (2.3)$$

assumes a stationary value at $\bar{\boldsymbol{\Lambda}}^0 = \{\bar{\mathbf{u}}^0, \bar{\boldsymbol{\gamma}}^0, \bar{\boldsymbol{\sigma}}^0\}$.

Proof. Let $\hat{\bar{\boldsymbol{\Lambda}}} = \{\hat{\bar{\mathbf{u}}}, \hat{\bar{\boldsymbol{\gamma}}}, \hat{\bar{\boldsymbol{\sigma}}}\}$ denote an arbitrary element in the domain of K. Then using (2.4) or (2.5) of [1], we compute

$$\delta K(\bar{\boldsymbol{\Lambda}}) = \lim_{\alpha\to 0}\frac{1}{\alpha}[K(\bar{\boldsymbol{\Lambda}} + \alpha\hat{\bar{\boldsymbol{\Lambda}}}) - K(\bar{\boldsymbol{\Lambda}})].$$

Now comes a great deal of algebra. To illustrate a typical manipulation, consider the variation of the third term in (2.3). We have

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\frac{1}{4\pi j}\int_v \{[(\bar{\sigma}^{ij}+\alpha\hat{\sigma}^{ij}) * (\bar{u}_{m,j}+\alpha\hat{u}_{m,j})] * (\bar{u}_{m,i}+\alpha\hat{u}_{m,i}) - (\bar{\sigma}^{ij} * \bar{u}_{m,j}) * \bar{u}_{m,i}\}\,dv$$

$$=\lim_{\alpha\to 0}\frac{1}{\alpha}\frac{1}{4\pi j}\int_v \{\alpha(\hat{\sigma}^{ij} * \bar{u}_{m,j} * \bar{u}_{m,i} + \bar{\sigma}^{ij} * \hat{u}_{m,j} * \bar{u}_{m,i} + \bar{\sigma}^{ij} * \bar{u}_{m,j} * \hat{u}_{m,i}) + 0(\alpha^2)\}\,dv$$

$$=\frac{1}{4\pi j}\int_v \{(\bar{u}_{m,i} * \bar{u}_{m,j}) * \hat{\sigma}^{ij} + 2(\bar{\sigma}^{ij} * \bar{u}_{m,j}) * \hat{u}_{m,i}\}\,dv$$

$$=\frac{1}{4\pi j}\int_v \{(\bar{u}_{m,i} * \bar{u}_{m,j}) * \hat{\sigma}^{ij} - 2(\hat{\sigma}^{ij} * \bar{u}_{m,j})_{,i} * \hat{u}_m\}\,dv.$$

Performing similar calculations with the remaining terms in (2.3) and collecting the results, we get

$$\delta K(\Lambda) = \int_v \left\{ \left[\rho p^2 \bar{u}_m - \frac{1}{2\pi j}(\bar{\sigma}^{ij} * \bar{u}_{m,j})_{,i} - \bar{\sigma}^{im}_{,i} - \rho \bar{F}_m \right] * \hat{u}_m \right.$$
$$\left. - \left(\bar{\gamma}_{ij} - \tfrac{1}{2}\left[\bar{u}_{i,j} + \bar{u}_{j,i} + \frac{1}{2\pi j}(\bar{u}_{m,i} * \bar{u}_{m,j}) \right] \right) * \hat{\sigma}^{ij} - (\bar{\sigma}^{ij} - F^{ij}[\bar{\gamma}(p)]) * \hat{\gamma}_{ij} \right\} dv$$
$$= \langle \text{grad } K(\bar{\Lambda}), \hat{\Lambda} \rangle.$$

Clearly grad $K(\bar{\Lambda}) = 0$ at $\bar{\Lambda} = \bar{\Lambda}^0$, which proves the theorem.

3. AN EXAMPLE

Obviously, theorem 1 holds only if functionals of the type $\psi[\cdot]$ in (2.3) exist. We offer an example in which such a functional does indeed exist. Consider a class of viscoelastic materials which, for one-dimensional deformations, the stress is given by

$$\sigma(t) = \int_0^t G_1(t-\tau)\frac{\partial\gamma}{\partial\tau}(\tau)\,d\tau + \int_0^t\int_0^t G_2(t-\tau, t-\eta)\frac{\partial\gamma}{\partial\tau}(\tau)\frac{\partial\gamma}{\partial\eta}(\eta)\,d\tau\,d\eta$$
$$+ \int_0^t\int_0^t\int_0^t G_3(t-\tau, t-\eta, t-\xi)\frac{\partial\gamma}{\partial\tau}(\tau)\frac{\partial\gamma}{\partial\eta}(\eta)\frac{\partial\gamma}{\partial\xi}(\xi)\,d\tau\,d\eta\,d\xi \tag{3.1a}$$

or equivalently

$$\sigma(t) = G_1 * \dot{\gamma} + G_2 * \dot{\gamma} * \dot{\gamma} + G_3 * \dot{\gamma} * \dot{\gamma} * \dot{\gamma} \tag{3.1b}$$

where G_1, G_2 and G_3 are material kernels and integral representations such as these are common in nonlinear viscoelasticity; see, for example [2].

Now taking the Laplace transform of (3.1), and assuming, for simplicity, homogeneous initial conditions, we obtain

$$\bar{\sigma}(p) = F[\bar{\gamma}(p), p] = \bar{G}_1 p\bar{\gamma}(p) + \bar{G}_2 p^2\bar{\gamma}(p)\bar{\gamma}(p) + \bar{G}_3 p^3\bar{\gamma}(p)\bar{\gamma}(p)\bar{\gamma}(p) \tag{3.2}$$

wherein again the dependences of $\bar{\sigma}$ and $\bar{\gamma}$ on $\mathbf{x}$ are understood. Then, following the procedure described in [1], we compute the integral

$$J \equiv \int_0^1 \langle F[\lambda\bar{\gamma}(p), p], \bar{\gamma}(p)\rangle\,d\lambda$$
$$= \int_v \{\tfrac{1}{2}\bar{G}_1 p\bar{\gamma} * \bar{\gamma} + \tfrac{1}{3}\bar{G}_2 p^2(\bar{\gamma})^2 * \bar{\gamma} + \tfrac{1}{4}\bar{G}_3 p^3(\bar{\gamma})^3 * \bar{\gamma}\}\,dv$$
$$= \int_v \Psi[\bar{\gamma}(p), p]\,dv. \tag{3.3}$$

Now for the functional $\psi[\cdot]$ in (2.3) to possess the desired properties, we assume that the variable $\bar{\gamma}(\mathbf{x}, p)$ be represented as a product of a function of $\mathbf{x}$ and one of p; i.e.

$$\bar{\gamma}(\mathbf{x}, p) \equiv \bar{\bar{\gamma}}(\mathbf{x}) f(p). \tag{3.4}$$

Using (3.4) in (3.3) and considering variations in $\bar{\bar{\gamma}}_{ij}(\mathbf{x})$ only, we can verify that

$$\delta J = \int_v \{\bar{G}_1 p \bar{\gamma} + \bar{G}_2 p^2 (\bar{\gamma})^2 + \bar{G}_3 p^3 (\bar{\gamma})^3\} * \hat{\bar{\gamma}} \, dv \tag{3.5a}$$

or

$$\delta J = \int_v F[\bar{\gamma}(p), p] * \hat{\bar{\gamma}}(p) \, dv = \int_v \delta\psi[\bar{\gamma}(p)] \, dv. \tag{3.5b}$$

4. REMARKS

It should be noted that for the case where the constitutive equation (3.1) is a linear relation, separability property of (3.4) is not required. In our example, it is the material nonlinearity (as described in (3.1)) which compels us to impose this condition.

We also remark that equation (3.9d) in [1] for the internal dissipation be deleted from the system of field equations and the remaining equations be written in the transformed form such as given in (2.2) of the present note. Then the order of the operator matrix appearing in (4.1) reduces to three.

The terms appearing in equations (5.1–5.3) of [1] should be replaced by the corresponding transformed terms of (2.4) of this note and the quantity $\gamma_{ij} * \mathfrak{F}^{ij}[\cdot]$ should be replaced by the functional $\psi[\cdot]$ described herein. In equations (5.6–5.7) introduce ρ in front of the free energy functional Φ and replace $\hat{\sigma}$ and $\delta\hat{\sigma}$ by $D(\mathbf{u}, \hat{\sigma})$ and $\delta D(\cdot)$ respectively; where D is defined so that (5.8) is $\delta_{\mathbf{u}} D = \hat{\sigma} = \rho \delta_{\boldsymbol{\gamma}} \Phi_{s=0}^{\infty} [\boldsymbol{\gamma}^t(\mathbf{u}); \boldsymbol{\gamma}(\mathbf{u}) \mid (\partial\gamma/\partial\mathbf{u}) \, \dot{\mathbf{u}}^t]$.

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